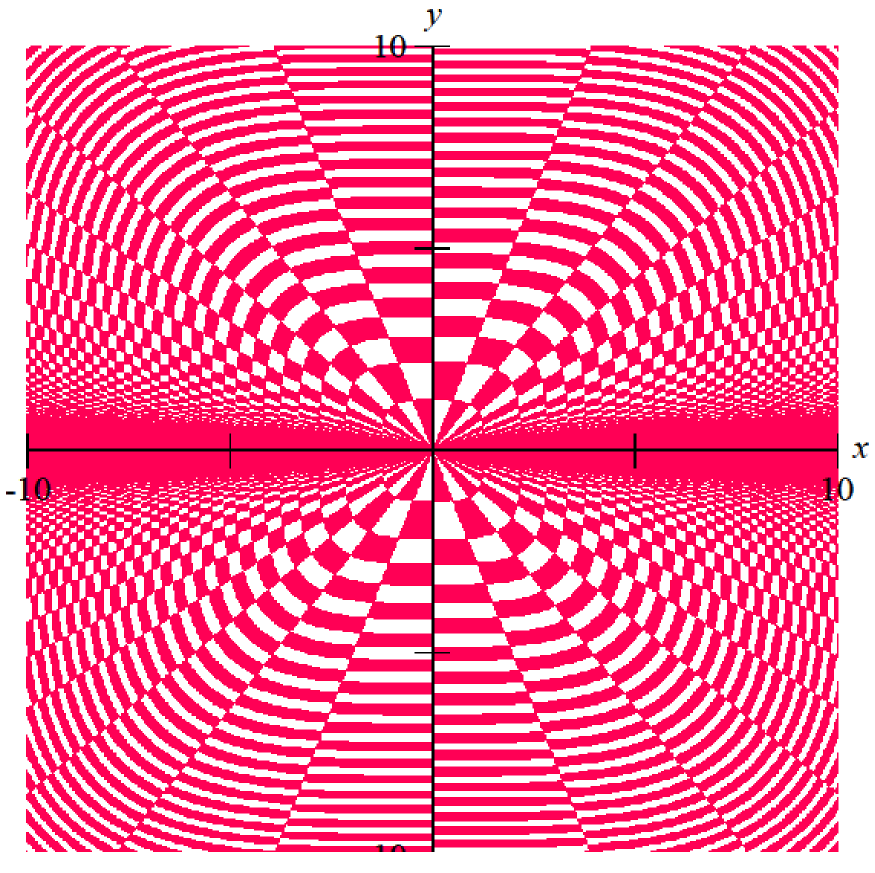
WELLINGTON COLLEGE
MATHEMATICS DEPARTMENT

Sixth Form Kick Start



$$\cos\left(\sqrt{x^4+y^4}\right)\sin\left(\frac{8x}{y}\right)<0$$

Introduction

There is a big step up from IGCSE to AS-Level or IB: questions are longer; there is a non-calculator paper; you are given less guidance. Additionally, you will discover that there is a change in the way you are expected to work: your teachers will expect you to work independently; you will be given fewer drill exercises and more challenging problems. You will need greater tenacity and a real desire to persevere.

Those students who work well – and by which we mean who work hard – find AS Mathematics a rewarding and enjoyable course. Others, however, fall by the wayside. This is often because their algebraic skills are weak at the start of the course and they then do not make the effort required to strengthen their skills.

This booklet contains a revision of all the IGCSE material that you need to have mastered before starting the course. There is an emphasis here on being able to complete basic algebra accurately. If you can't do this confidently already, you need to practice the material in this booklet until you understand all the techniques it contains – and can apply them without error.

You will be tested on this material early in Lower Sixth. If you cannot complete it all accurately by then, you are probably not cut out for the course. We suggest you use this book to help you prepare. We will make answers available on the Maths Department Intranet site if you want to check your answers.

This book contains explanations for everything you need, but they are explanations designed to help you revise, not to start from scratch. If you are stuck, you may need a little extra help. We suggest the following sources:

- www.khanacademy.org has a tremendous selection of videos and exercises to help you practice and is completely free.
- www.mymaths.co.uk is a service to which the school has a subscription. You can log in using the school id "wellcoll" and our current password "hexagon". It contains online lessons and exercises which will help you practice.
- CGP revision guides. Many of you will already have some sort of GCSE/IGCSE revision guide. You can refer back to this if you are having difficulty.

We will be offering workshops in the first couple of weeks in September to help you get on top of this material if you are stuck, but the onus is upon you to seek help. Use this book over the summer to work out what you can and cannot do, and then use the workshops to fill in the gaps.

Mastering this material early will enable you to make much better progress through the sixth form. Use this opportunity wisely to get yourself ready.

Good luck

Mr Sproat

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Number

You need to be able to

- Identify the Natural Numbers, Integers, Rational Numbers, Irrational Numbers.
- Convert fractions to decimals
- Convert decimals to fractions

Types of number

The **Natural Numbers** are the counting numbers {1, 2, 3, 4, 5…}. You may also know them as **positive integers**. They do not include 0.

The **Integers** are all the whole numbers, including the negative numbers, e.g. $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. They include 0.

The **Rational Numbers** are all numbers which can be written as fractions. Anything which has a terminating decimal, or a repeating decimal, can be written as a fraction and so is rational. Zero is rational, because $0=\frac{0}{1}$, so it can be written as a fraction.

The **Irrational Numbers** are all the numbers on the number line which cannot be written as a fraction. There are very many different types of irrational number, but some examples include $\sqrt{2}, \pi, \frac{\sqrt{5}+1}{2}$.

The **Real Numbers** are all the numbers, rational or irrational, which you can find on a number line. All the numbers you have ever used are Real Numbers. We give them a name because there are some numbers used by mathematicians which do not fit anywhere on the number line. You will not meet these unless you study Further Mathematics.

Converting fractions to decimals

The horizontal line in a fraction represents division, so to express $\dfrac{1}{6}$ as a decimal we can complete the division: $1 \div 6$. You might lay it out as follows:

$$\begin{array}{r} 0.\ 1\ \ 6\ \ 6\ \ 6\ \ \ldots \\ 6\overline{)1.\ {}^{1}0\ {}^{4}0\ {}^{4}0\ {}^{4}0\ \ \ldots} \end{array}$$

And give your answer as $0.1\dot{6}$

If you are not sure how to do this, watch this video:

https://www.khanacademy.org/math/arithmetic/decimals/decimal_to_fraction/v/converting-fractions-to-decimals

Converting decimals to fractions – Terminating decimals

If a decimal terminates then we can convert it easily. Remember that the digits after the decimal point represent tenths, hundredths and thousandths, so $0.842=\dfrac{842}{1000}$. We can simplify this to give $\dfrac{421}{500}$, which would be our final answer.

Converting recurring decimals to fractions.

If a decimal repeats, e.g. $0.2727272727\ldots = 0.\overline{27}$, you can write it as a fraction by applying the following method:

Let $x = 0.27272727\ldots$ Therefore $100x = 27.27272727\ldots$ Subtracting the first equation from the second gives us $99x = 27$, so $x=\dfrac{27}{99}=\dfrac{3}{11}$. Therefore $0.\overline{27}=\dfrac{3}{11}$. If the repeating section had been only one digit long, we would have multiplied by 10; if it had been three digits long, we would have multiplied by 1000. The repeating section must always line up with itself when you have multiplied by the appropriate power of 10.

Exercises

1. For each of the following numbers, decide whether the given number is Rational or Irrational.

(a) 0.7 (b) π (c) $\sqrt{2}+1$ (d) $0.\overline{4}$

(e) 0 (f) $\dfrac{\sqrt{3}}{2}$ (g) 0.307307... (h) 0.125

2. Write each of the following fractions as decimals without using a calculator.

(a) $\dfrac{3}{4}$ (b) $\dfrac{3}{11}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{3}$

(e) $\dfrac{2}{9}$ (f) $\dfrac{3}{7}$ (g) $\dfrac{14}{15}$ (h) $\dfrac{5}{37}$

(i) $\dfrac{1}{8}$ (j) $\dfrac{13}{9}$ (k) $\dfrac{3}{5}$ (l) $\dfrac{3}{16}$

(m) $\dfrac{6}{5}$ (n) $\dfrac{33}{10}$ (o) $\dfrac{8}{15}$ (p) $\dfrac{4}{3}$

3. Write each of the following decimals as fractions in their simplest form without using a calculator.

(a) 0.3 (b) 0.456 (c) 0.212121… (d) 1.4

(e) 0.3333… (f) 0.104104… (g) 0.5555… (h) 0.125

(i) 0.1625 (j) 0.101010… (k) 0.363363… (l) 0.00325

Fractions

You need to be able to

- Convert top-heavy fractions to mixed-numbers without a calculator
- Convert mixed numbers to top-heavy fractions without a calculator
- Add fractions without a calculator
- Subtract fractions without a calculator
- Multiply fractions without a calculator
- Divide fractions without a calculator

Top heavy fractions

Mixed numbers are numbers which look like $1\frac{1}{2}$ or $3\frac{3}{7}$. They have an integer part and a fractional part.

Top-heavy fractions are just written with a fractional part, although the **numerator** (the number on top) may be bigger than the **denominator** (number on the bottom). It is easier to work with top-heavy fractions. To convert mixed numbers to top-heavy fractions, write the integer part as if it were a fraction and then add.

For example, $1\frac{1}{2} = 1 + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$. Similarly, $3\frac{3}{7} = 3 + \frac{3}{7} = \frac{21}{7} + \frac{3}{7} = \frac{24}{7}$.

To go the other way, find the largest multiple of the denominator which can be taken out of the numerator, and turn this into a whole number part.

For example, $\frac{23}{5} = \frac{20}{5} + \frac{3}{5} = 4 + \frac{3}{5} = 4\frac{3}{5}$.

Adding and subtracting

To add and subtract fractions, they must first be put over a common denominator.

For example $\frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$, and $\frac{6}{7} - \frac{1}{9} = \frac{54}{63} - \frac{7}{63} = \frac{47}{63}$.

Multiplying and dividing

To multiply fractions, multiply the top lines and the bottom lines together. Don't forget to cancel common factors if you can.

For example $\frac{14}{33} \times \frac{3}{35} = \frac{14\times 3}{33\times 35} = \frac{14\times 1}{11\times 35} = \frac{2\times 1}{11\times 5} = \frac{2}{55}$. You may well decide to write this using fewer steps.

To divide fractions, you have to multiply by the reciprocal of the fraction on the bottom. The reciprocal of a fraction is that fraction written upside down.

For example $\frac{2}{9} \div \frac{4}{3} = \frac{2}{9} \times \frac{3}{4} = \frac{1}{6}$.

Exercises

4. Convert the following mixed numbers to top-heavy fractions without using a calculator.

(a) $3\frac{2}{5}$ (b) $7\frac{1}{4}$ (c) $6\frac{1}{4}$

(d) $-2\frac{3}{7}$ (e) $8\frac{5}{6}$ (f) $6\frac{17}{24}$

5. Convert the following top-heavy fractions to mixed numbers without using a calculator.

(a) $\frac{11}{5}$ (b) $\frac{224}{15}$ (c) $\frac{132}{23}$

(d) $\frac{47}{3}$ (e) $\frac{64}{11}$ (f) $\frac{1047}{107}$

6. Evaluate the following without using a calculator, giving your answer as top-heavy fractions in their lowest terms.

(a) $\frac{3}{5}+\frac{12}{5}$ (b) $\frac{3}{4}+\frac{9}{8}$ (c) $\frac{1}{7}-\frac{1}{30}$

(d) $\frac{15}{4}-\frac{5}{6}$ (e) $\frac{13}{12}+\frac{1}{8}-\frac{2}{5}$ (f) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$

7. Evaluate the following without using a calculator, giving your answer as top-heavy fractions in their lowest terms.

(a) $\frac{3}{5}\times\frac{12}{5}$ (b) $\frac{3}{4}\div\frac{9}{8}$ (c) $\frac{2}{7}\times\frac{11}{30}$

(d) $\frac{15}{4}\div\frac{5}{6}$ (e) $\left(\frac{13}{12}\times\frac{1}{8}\right)\div\frac{2}{5}$ (f) $\frac{1}{2}\div\left(\frac{2}{3}\div\frac{3}{4}\right)$

The Laws of Indices

You need to be able to

- Use the laws of indices
- Evaluate negative indices without a calculator.
- Evaluate fractional indices without a calculator.

Rules

You must learn the following and be able to apply them

$a^m \times a^n = a^{m+n}$ $\qquad \dfrac{a^m}{a^n} = a^{m-n}$ $\qquad (a^m)^n = a^{mn}$

$a^{-1} = \dfrac{1}{a}$ $\qquad a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{1}{n}} = \sqrt[n]{a}$ $\qquad a^{\frac{p}{q}} = (\sqrt[q]{a})^p$ $\qquad a^0 = 1$

You need to be able to use the rules to simplify expressions without a calculator.

For example, $125^{\frac{-2}{3}} = \dfrac{1}{125^{\frac{2}{3}}} = \dfrac{1}{(\sqrt[3]{125})^2} = \dfrac{1}{5^2} = \dfrac{1}{25}$, and $\dfrac{4}{\sqrt[3]{2}} = \dfrac{2^2}{2^{\frac{1}{3}}} = 2^{2-\frac{1}{3}} = 2^{\frac{5}{3}}$.

You need to be able to use these rules to write numbers in a given form. For example,

Q: Write $\dfrac{9^5}{\sqrt{3}}$ in the form 3^x.

A: $\dfrac{9^5}{\sqrt{3}} = \dfrac{\left(3^2\right)^5}{3^{\frac{1}{2}}} = \dfrac{3^{10}}{3^{\frac{1}{2}}} = 3^{10-\frac{1}{2}} = 3^{\frac{19}{2}}$.

Exercises

8. Write the following in the form 2^n.

(a) 4^3 (b) 16^{-1} (c) 256^0

(d) $\frac{1}{8^{-2}}$ (e) $\frac{64^2}{8}$ (f) $\frac{4^2}{8^3}$

(g) $\frac{6^3}{27}$ (h) $\frac{4^3 \times 8^{-2}}{16}$

9. Evaluate without using a calculator

(a) $\left(27^{\frac{1}{3}}\right)^2$ (b) $\left(4^3\right)^{\frac{1}{2}}$ (c) $\left(10000^{\frac{1}{4}}\right)^{-1}$

(d) $\left(8^{-2}\right)^{\frac{1}{3}}$ (e) $\left(81^{\frac{1}{2}}\right)^{-2}$ (f) $\left[\left(\frac{1}{8}\right)^2\right]^{-\frac{1}{3}}$

10. Evaluate without using a calculator

(a) $8^{\frac{2}{3}}$ (b) $16^{\frac{3}{4}}$ (c) $1000^{\frac{2}{3}}$

(d) $25^{\frac{3}{2}}$ (e) $27^{-\frac{2}{3}}$ (f) $16^{\frac{3}{2}}$

(g) $9^{\frac{-3}{2}}$ (h) $64^{\frac{2}{3}}$ (i) $\left(\frac{1}{8}\right)^{-\frac{5}{3}}$

11. Find the prime factorisations of the following.
(a) 32
(b) 216
(c) 729
(d) 81
(e) 64

12. Using your answers to question 8, evaluate the following without using a calculator,

(a) $32^{\frac{2}{5}}$ (b) $32^{\frac{4}{5}}$ (c) $81^{\frac{3}{4}}$ (d) $81^{\frac{5}{4}}$

(e) $216^{\frac{2}{3}}$ (f) $216^{\frac{4}{3}}$ (g) $64^{\frac{2}{3}}$ (h) $64^{\frac{7}{6}}$

(i) $64^{\frac{3}{2}}$ (j) $729^{\frac{1}{3}}$ (k) $729^{\frac{5}{6}}$ (l) $729^{\frac{1}{2}}$

Surds

You need to be able to

- Use and manipulate surds without a calculator
- Convert surds to index form
- Rationalise the denominator where it is of the form $\sqrt{n}$.

Simplifying surds

Most of the manipulation of surds comes from using the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$, either forwards or backwards.

We generally prefer surds to have the smallest number under the square roots sign that is possible. For example, $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$. This surd has been simplified. This is useful when simplifying expressions using surds, for example $\sqrt{12} + \sqrt{48} = \sqrt{4}\sqrt{3} + \sqrt{16}\sqrt{3} = 2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$.

Rationalising the denominator

Fractions with surds in the denominator can be problematic. We can rationalise the denominator by multiplying it by something which will make it rational. If the denominator is just a square root, then multiplying it by itself will do the trick. Remember to multiply the numerator by the same thing.

For example, $\frac{10}{\sqrt{5}} = \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$.

Surds and index form

In the last section you learnt that $a^{\frac{p}{q}} = (\sqrt[q]{a})^p$. You may need to write a surd in index form. For example,

Q: Write $\sqrt[3]{9} \times \sqrt{3}$ as a power of 3.

A: $\sqrt[3]{9} \times \sqrt{3} = \sqrt[3]{3^2} \times \sqrt{3} = 3^{\frac{2}{3}} \times 3^{\frac{1}{2}} = 3^{\frac{2}{3}+\frac{1}{2}} = 3^{\frac{7}{6}}$.

Examples

13. Simplify the following without using a calculator.
    (a) $\left(3+\sqrt{5}\right)+\left(2+\sqrt{5}\right)$
    (b) $\left(1-\sqrt{2}\right)-\left(3-2\sqrt{2}\right)$
    (c) $\left(1+\sqrt{3}\right)-\left(1-\sqrt{5}\right)$
    (d) $\left(\sqrt{3}+3\sqrt{5}\right)-\left(2\sqrt{3}-\sqrt{5}\right)$
    (e) $\left(1+\sqrt{2}+4\sqrt{3}\right)-\left(-2\sqrt{3}+2\sqrt{2}\right)$

14. Simplify the following without using a calculator.
    (a) $2\sqrt{3}$
    (b) $\sqrt{3}\times\sqrt{2}$
    (c) $\sqrt{3}+\sqrt{3}+\sqrt{3}$
    (d) $\sqrt{48}\times\sqrt{3}$
    (e) $\sqrt{5}+\sqrt{5}+\sqrt{20}$
    (f) $3\sqrt{\frac{10}{3}}\times\sqrt{\frac{5}{6}}$

15. Expand and simplify without using a calculator.
    (a) $2\left(1+\sqrt{2}\right)$
    (b) $\left(1-\sqrt{7}\right)\left(\sqrt{7}+3\right)$
    (c) $\sqrt{2}\times\sqrt{5}$
    (d) $\left(3-\sqrt{2}\right)\left(3+\sqrt{2}\right)$
    (e) $\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)$
    (f) $\sqrt{3}\left(\sqrt{3}-1\right)$

16. Write the following as a power of 2.
    (a) $8\sqrt{2}$
    (b) $4\sqrt[3]{2}$
    (c) $(\sqrt{2})^4$
    (d) $\frac{16}{\sqrt{2}}$
    (e) $\frac{\sqrt[3]{2}}{4}$
    (f) $\frac{4^2}{(\sqrt{2})^3}$

Solving linear equations

You need to be able to

- Solve linear equations, including those where the subject appears more than once

Linear equations are those where the unknown has not been raised to a power.

When solving equations, it is vital that you remember the golden rule of algebra:

Whatever is done to one side must also be done to the other.

If you have some notion of moving things from one side to another, and signs changing, or multiplication becoming division, or anything else like that: abandon it. Simply remember:

Whatever is done to one side must also be done to the other.

Nothing else is reliable.

In equations where the variable appears more than once, or in brackets, you need to collect all the instances of the variable in one place. This is usually most easily done by multiplying out the brackets.

It is very easy to check if you have solved an equation correctly by substituting your answer back into the original equation.

Examples

Q: Solve $8(x+3)=3(x+7)+13$

A: $8(x+3)=3(x+7)+13$

$8x+24=3x+21+13$

$5x=20$

$x=4$

Q: Solve $\frac{1}{2}(3x-7)-2(x-3)=2x$

A: $\frac{1}{2}(3x-7)-2(x-3)=2x$

$(3x-7)-4(x-3)=4x$

$3x-7-4x+12=4x$

$5=5x$

$x=1$

Exercises

17. Solve without using a calculator, leaving answers as top-heavy fractions where necessary.
    (a) $3x+7=15$
    (b) $13(x-4)=2x$
    (c) $9(x-2)-3(4-2x)=7$
    (d) $10(x-2)-3(x+7)=0$
    (e) $4x+11=5(x-3)$
    (f) $\frac{1}{3}(x-3)+2(x-5)=2x+7$
    (g) $\dfrac{2x-4}{5}-\dfrac{3}{4}(x+2)=1$
    (h) $7(x-2)-3(7x-3)+\dfrac{x+5}{6}=3(x-8)$
    (i) $14(11x+9)-12(8x-2)=11(3x-8)$

Solving quadratic equations

You need to be able to

- solve quadratic equations by factorising.
- solve quadratic equations using the quadratic formula, which you must memorise.
- solve quadratic equations that are presented in other forms.

Factorising

Quadratics can be solved by factorising. Remember that the quadratic expression must be equal to zero first. The important detail is that if two things multiply together to give zero then one or other of them must be zero.

For example

Q: Solve $x^2+3=4x$

A: We rearrange and then factorise.

$x^2-4x+3=0$

$(x-3)(x-1)=0$

So either $x-3=0$ or $x-1=0$

So $x=1$ or $x=3$.

Q: Solve $x(6x+1)=2$

A: We rearrange and then factorise.

$6x^2+x-2=0$

$(2x-1)(3x+2)=0$

So either $2x-1=0$ or $3x+2=0$

So $x=\frac{1}{2}$ or $x=-\frac{2}{3}$.

The quadratic formula

If $ax^2+bx+c=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. You must memorise this formula.

For example

Q: Solve $3x^2-7x+1=0$

A: $a=3, b=-7, c=1$

$$x=\frac{7\pm\sqrt{49-4\times3\times1}}{6}$$

$$x=\frac{7\pm\sqrt{37}}{6}$$

$x=0.153$ or $x=2.18$

Q: Solve $11.2x^2-4.3x-2=0$

A: $a=11.2, b=-4.3, c=-2$

$$x=\frac{4.3\pm\sqrt{(-4.3)^2-4\times11.2\times(-2)}}{22.4}$$

$x=-0.2722$ or $x=0.6561$

Quadratics in disguise

Some equations, especially those with algebraic fractions in them, rearrange to give you a quadratic equation.

For example

Q: Solve $\dfrac{3}{x+3}=x+2$

A: Multiplying through by $(x+3)$ gives $3=(x+2)(x+3)$ and so

$3=x^2+5x+6$

$0=x^2+5x+3$

$$x=\frac{-5\pm\sqrt{25-12}}{2}=\frac{-5\pm\sqrt{13}}{2}\approx-4.30 \text{ or } -0.70.$$

Exercises

18. Solve the following equations.

    (a) $x^2 + 7x + 12 = 0$

    (b) $x^2 + 6x - 27 = 0$

    (c) $x^2 + 9x = 22$

    (d) $x^2 + 10 = 7x$

    (e) $2x^2 + 15x + 7 = 0$

    (f) $2x^2 - 9x - 18 = 0$

    (g) $(3x - 8)(x - 1) = 0$

    (h) $(3x - 8)(x - 1) = 12$

    (i) $x(4x - 5) = 6$

19. Solve the following equations using the quadratic equation formula, leaving your answers as surds in their simplest form where appropriate. Write out the formula in full each time.

    (a) $3A^2 - 8A + 5 = 0$

    (b) $2\beta^2 + 3\beta - 4 = 0$

    (c) $5C^2 + 6C - 2 = 0$

    (d) $3d^2 - 4d + 1 = 0$

    (e) $5e^2 - 2e - 6 = 0$

    (f) $4f\left(f + 3\right) = 10$

    (g) $\left(g + 4\right)\left(2g - 1\right) = 3$

    (h) $2 + 3h - h^2 = 0$

20. Solve these equations.

    (a) $\frac{x+1}{x-1} = \frac{2x+5}{x+1}$

    (b) $\frac{x+3}{x+1} = \frac{2x+3}{x+3}$

    (c) $\frac{x+1}{x-1} = \frac{2x+8}{x+2}$

    (d) $\frac{x}{2x+3} + 1 = x$

Simple trigonometric equations

You need to be able to
- solve simple equations involving sine, cosine and tangent function
- recognise how to use the inverse trigonometric functions, arcsin, arccos and arctan.

Trigonometric functions and their inverse functions

Most of the functions you have encountered can be 'undone'. This means that the function $f(x)$ has an inverse function $f^{-1}(x)$. This is true of the trigonometric functions – with a few extra details which you will learn about later in the course.

For the time being you can use the fact that $\arcsin(x)$, sometimes written $\sin^{-1}(x)$, is the inverse of $\sin(x)$. The same goes for the other two functions. This means that $\arcsin(\sin(x)) = x$, so if you 'do' arcsin to $\sin(x)$, you're just left with *x*.
Remember that whatever is done to one side of an equation must be done to the other.

For example

Q: Solve $\sin x = 0.6$
A: We apply arcsin to both sides

$$\begin{aligned}\arcsin(\sin(x)) &= \arcsin 0.6\\ x &= \arcsin 0.6\\ x &= 36.9^\circ\end{aligned}$$

Q: Solve $\tan(x+30) = 1$
A: We apply arctan to both sides

$$\begin{aligned}\arctan(\tan(x+30)) &= \arctan 1\\ x+30 &= 45\\ x &= 15^\circ\end{aligned}$$

Similarly, you can use sin to get rid of arcsin.

For example

Q: Solve $\arcsin x = 60$
A: We apply sin to both sides

$$\begin{aligned}\arcsin x &= 60\\ \sin(\arcsin x) &= \sin 60\\ x &= 0.866\end{aligned}$$

The trigonometric expression could be part of a more complicated equation.

For example

Q: Solve $3\sin x + 4 = \sin x + 3$
A: We rearrange and solve.

$$\begin{aligned}3\sin x + 4 &= \sin x + 3\\ 2\sin x &= -1\\ \sin x &= -\tfrac{1}{2}\\ x &= \arcsin\left(-\tfrac{1}{2}\right)\\ x &= -30^\circ\end{aligned}$$

Q: Solve $\dfrac{3\tan x + 1}{4 - \tan x} = 5$
A: We rearrange and solve.

$$\begin{aligned}\frac{3\tan x + 1}{4 - \tan x} &= 5\\ 3\tan x + 1 &= 5(4 - \tan x)\\ 3\tan x + 1 &= 20 - 5\tan x\\ 8\tan x &= 19\\ \tan x &= \tfrac{19}{9}\\ x &= \arctan\tfrac{19}{8}\\ x &= 67^\circ\end{aligned}$$

Exercises

21. Solve the following equations.
Give your answers correct to one decimal place.
    (a) $\sin x = 0.6$
    (b) $\cos x = -0.3$
    (c) $\tan x = -0.2$
    (d) $\sin x = -0.45$
    (e) $\tan x = 5.7$

22. Solve the following equations, giving your answers correct to the nearest degree.
    (a) $2\cos x = 0.3$
    (b) $5\tan x - 3 = 2\tan x + 11$
    (c) $2(\sin x + 1) - 5(\sin x - 2) = -6$

23. Solve the following equations, giving your answer correct to three decimal places.
    (a) $\arcsin(3(\tan x - 2)) = 25$
    (b) $\arccos(x - 5) = 60$
    (c) $\sqrt{\sin(x - 30)} = 0.6$

Linear simultaneous equations

You need to be able to

- Solve linear simultaneous equations by substitution.
- Solve linear simultaneous equations by elimination.

Substitution

In substitution, we rearrange one equation so that one of the unknowns is the subject.

For example,

Q: Solve $\begin{cases} x + 2y = 7 \\ 3x + y = 4 \end{cases}$

A: We choose one equation and rearrange it to make one of the unknowns the subject.
For example, we could make *x* the subject of the top equation:

$$x = 7 - 2y$$

We then substitute into the second equation:

$$3(7 - 2y) + y = 4$$

and solve the equation that results:

$$\begin{aligned} 3(7 - 2y) + y &= 4 \\ 21 - 6y + y &= 4 \\ 17 &= 5y \\ y &= 3.4 \end{aligned}$$

We can then find the value of the other variable by substituting the value we have found into an earlier equation:

$$\begin{aligned} x &= 7 - 2y \\ x &= 7 - 2 \times 3.4 \\ x &= 0.2 \end{aligned}$$

Elimination

In elimination, a multiple of one equation is added to or subtracted from the other in order to remove an unknown.

For example,

Q: Solve $\begin{cases} x + 2y = 7 \\ 3x + y = 4 \end{cases}$

A: Doubling the second equation gives $6x + 2y = 8$. So, subtracting the first equation from our new one gives:

$$\begin{array}{rr} & 6x + 2y = 8 \\ - & x + 2y = 7 \\ \hline & 5x = 1 \end{array}$$

So we have $x = 2$. The *y* value can be found by substituting back into an equation, just as above.

Exercises

24. Solve the following pairs of simultaneous equations by substitution

(a) $x + y = 6$ $2x + y = 8$

(b) $x + 2y = 8$ $3x + y = 9$

(c) $a + 2b = 11$ $2a + b = 13$

(d) $7p + 2q = 19$ $p - q = 4$

(e) $2k + 7j = 17$ $5k + 3j = -1$

(f) $3x - y = 17$ $\frac{x}{3} + \frac{y}{2} = 0$

25. Solve the following pairs of simultaneous equations by elimination.

(a) $2x + 3y = 8$, $3x + 2y = 7$

(b) $3x + 5y = 20$, $x + 4y = 16$

(c) $5x - 2y = 3$, $x + 4y = 5$

(d) $2a - b = 5$, $\frac{1}{4}a + \frac{1}{3}b = 2$

(e) $2p + 3q = 11$, $\frac{1}{2}p + \frac{2}{3}q = \frac{5}{2}$

(f) $2x + 5y = 24$, $x + \frac{3}{4}y = 5$

(g) $3s - 2t = 4$, $\frac{2}{3}s + \frac{1}{2}t = -\frac{7}{9}$

(h) $2u + v = 11$, $\frac{1}{5}u - \frac{1}{4}v = 1$

(i) $\frac{2}{5}l + 3m = 2.6$, $l - 2m = 4.6$

Non-linear simultaneous equations

You need to be able to:

- Solve non-linear simultaneous equations by substitution

Substitution

The method of substitution is no different to when solving linear simultaneous equations, expect the resulting equation will be harder to solve: probably quadratic. The most common errors come from algebraic mistakes, particularly when multiplying out brackets.

For example

Q: The circle *C* has equation $x^2 + y^2 = 25$, and the straight line *L* has the equation $x + 2y = 10$. Find the points of intersection of *C* and *L*

A: First we need to rearrange one equation. Since the first equation is full of squared terms, we'll use the easier one:

$$x + 2y = 10 \Rightarrow x = 10 - 2y$$

Substituting for *x* in the other equation yields:

$$(10 - 2y)^2 + y^2 = 25.$$

Squaring (carefully) gives:

$$\begin{aligned} 100 - 40y + 4y^2 + y^2 &= 25 \\ 5y^2 - 40y + 75 &= 0 \\ y^2 - 8y + 15 &= 0 \\ (y-3)(y-5) &= 0 \end{aligned}$$

So $y = 3$ or $y = 5$.

If $y = 3$, $x = 10 - 2 \times 3 = 4$. If $y = 5, x = 10 - 2 \times 5 = 0$.

Hence either $x = 4$, $y = 3$ or $x = 0, y = 5$.

Exercises

26. The parabola *P* has equation $y = x^2 - 12x + 40$, and the straight line *L* has equation $y = x + 4$.
Find the coordinates of all of the points of intersection.

27. The parabola *P* has equation $y = x^2 + 2x - 12$, and the straight line *L* has equation $y = x + 8$.
Find the coordinates of all of the points of intersection.

28. The circle *C* has equation $x^2 + (y - 2)^2 = 50$, and the straight line *L* has equation $y = x + 2$.
Find the coordinates of all of the points of intersection.

29. The hyperbola *H* has equation $xy = 7$, and the straight line *L* has equation $x + y = 6$.
Find the coordinates of all of the points of intersection.

30. The parabola *P* has equation $y = 3x^2 + 3x - 1$, and the straight line *L* has equation $y = 2x + 1$.
Find the coordinates of all of the points of intersection.

31. The curve *C* has equation $y = x(x + 1)(x - 1)$, and the hyperbola *H* has equation $xy = 12$.
Find the coordinates of all of the points of intersection.

Solving linear inequalities

You need to be able to:

- Solve inequalities by rearranging.
- Solve pairs of inequalities by synthesising the solution of two or more linear inequalities.

Rearranging

Solving inequalities is very similar to solving equations. Remember that when solving equations, the golden rule is that whatever is done to one side must be done to the other. With inequalities, the same holds true – but there is a shorter list of things that are permissible. With equations, you can do pretty much anything to both sides and it will be ok. Not with inequalities. Consider the following error:

$$3 < 5$$
$$\therefore\ -3 < -5$$

We have multiplied both sides by –1. Although this seems fine, you can see that it has turned a true statement into nonsense. As such there are some things that you can, and some that you cannot, do to inequalities.

Sometimes you may be given two inequalities to solve simultaneously. For these, solve each individually and find the region that satisfies both.

Things you can do

- Add to both sides
- Subtract from both sides
- Multiply both sides by a positive number
- Divide both sides by a positive number
- Cube or take the cube root of both sides

Things you can do if you are very careful

- Multiply or divide by a negative number.
  When you multiply or divide by a negative number, the inequality sign changes direction, so "greater than" becomes "less than" etc.
- Square or square root both sides.
  This is only possible if you know that the sides of your equation are both positive.

Things you mustn't do

- Multiply both sides by an unknown (in case it is negative)
- Apply trigonometric functions to both sides
- Take the reciprocal of both sides

For example

Q: Solve $4x + 7 < 2(3 - x)$

A: We start by simplifying each side and then rearrange.

$$4x + 7 < 2(3 - x)$$
$$4x + 7 < 6 - 2x$$
$$6x < 1$$
$$x < \tfrac{1}{6}$$

Q: Solve $-3x < 12$

A: We divide both sides by –3, remembering to change the sign.

$$-3x < 12$$
$$x > \frac{12}{-3}$$
$$x > -4$$

Exercises

32. Solve
    (a) $9x+7>5$
    (b) $4x-7(x+3)>x$
    (c) $x+4-4(x-4)<0$
    (d) $2x+4<5x-12$
    (e) $4x-7>5$
    (f) $3x-5>x$
    (g) $3x-7<7x-3$

33. Show that if $2x-5<9$ and $3+2x>9$ then $3<x<7$.

34. Solve simultaneously
    (a) $3x+1>0$ and $1+4x<11$
    (b) $2-3x<9$ and $5<8-x$
    (c) $3x+1>0$ and $5<8-x$
    (d) $2-3x<9$ and $3x+1>0$

Polynomials

You need to be able to

- Expand brackets
- Collect like terms
- Factorise
- Use the difference of two squares identity

Expanding brackets and collecting like terms

You will have already learnt a method for multiplying two brackets together (FOIL, 'crab claws'… there are several). What matters is that every term in the first bracket must be multiplied by every term in the other, even if the bracket has more than two terms in it. When you have multiplied out, 'like terms' are those that have the same structure – typically the same power of *x*. We group them together for tidiness.

For example

Q: Expand $(2x+3)(4x-5)$

A: Multiplying out gives:

$$\begin{aligned}&(2x+3)(4x-5)\\&=2x\times4x+3\times4x-5\times2x-15\\&=8x^2+12x-10x-15\\&=8x^2+2x-15\end{aligned}$$

Q: Expand $(x^2+x+1)(2x+3)$

A: Multiplying out gives:

$$\begin{aligned}&(x^2+x+1)(2x+3)\\&=2x^3+2x^2+2x+3x^2+3x+1\\&=2x^3+5x^2+5x+1\end{aligned}$$

Factorising

You should be able to factorise quadratic expressions – i.e. write them using brackets. The goal is to find two brackets which multiply together to give you a certain expression.
A special example is called 'the difference of two squares', and it enables you to factorise examples where there is no middle term. It uses the fact that $x^2-y^2=(x+y)(x-y)$, a fact which you can check by multiplying out the brackets.

For example

Q: Factorise $x^2-7x+12$

A: We need two brackets, and a pair of numbers that add to give 12 and which give us $-7x$ when we multiply out. Since the factor pairs of 12 are 1 & 12, 2 & 6, 3 & 4, and their negative pairs, we use

$$\begin{aligned}&x^2-7x+12\\&=(x-3)(x-4)\end{aligned}$$

Q: Factorise $4x^2-81$

A: We notice that $4x^2=(2x)^2$ and $81=9^2$ so we have the difference of two squares. Hence

$$\begin{aligned}&4x^2-81\\&=(2x+9)(2x-9)\end{aligned}$$

Exercises

35. Multiply out the following quadratic expressions.
    (a) $(a+2)(a+3)$
    (b) $(b-3)(b-4)$
    (c) $(c-5)(c+2)$
    (d) $d(d-4)$
    (e) $(e+3)(e-3)$

36. Multiply out these expressions by first multiplying together one pair of brackets.
    (a) $(x+2)(x-5)(x-3)$
    (b) $(x+2)(x-1)(x-2)$
    (c) $(x-4)^2(x-5)$
    (d) $(x+1)(x+2)(x+4)$

37. Factorise these quadratic expressions:
    (a) $b^2+2b-15$
    (b) $r^2-5r-14$
    (c) $y^2-8y+15$
    (d) $k^2+2k-63$
    (e) t^2-3t
    (f) w^2-16
    (g) $a^2+7a-30$
    (h) $n^2+13n-14$
    (i) r^2-5r
    (j) p^2-6p+5
    (k) t^2-36

38. Factorise these expressions:
    (a) $5r^2-15r$
    (b) $4z^2-9$
    (c) x^3+2x^2-3x
    (d) x^4-13x^2+36

Algebraic Fractions

You need to be able to

- Simplify expressions
- Solve equations involving algebraic fractions

Expressions

An algebraic fraction is any fraction where either the numerator or denominator is an algebraic expression and not simply a number, for example $\frac{1}{x+2}, \frac{x}{x-3}, \frac{2x^2+17x-9}{x^3-8}, \frac{3}{x}$.

Things to remember

- To multiply, simply multiply the tops together and the bottoms together.
- To divide one fraction by another, multiply by the reciprocal of the faction you are dividing by.
- To add or subtract fractions, you need a common denominator.
- Always try to simplify as much as possible by cancelling common factors. You will need to factorise first!

For example

Q: Simplify $\frac{x}{x+2} \div \frac{2x^2}{3x+6}$

A: First, we take the reciprocal of the second fraction.

$$\frac{x}{x+2} \div \frac{2x^2}{3x+6}$$
$$= \frac{x}{x+2} \times \frac{3x+6}{2x^2}$$

Then factorise the top line

$$\frac{x}{x+2} \times \frac{3(x+2)}{2x^2}$$
$$= \frac{1}{1} \times \frac{3}{2x} = \frac{3}{2x}$$

Q: Simplify $\frac{1}{x+3} + \frac{2x}{x-2}$

A: We choose $(x+3)(x-2)$ as a suitable denominator.

$$\frac{1}{x+3} + \frac{2x}{x-2}$$
$$= \frac{x-2}{(x+3)(x-2)} + \frac{2x(x+3)}{(x+3)(x-2)}$$
$$= \frac{x-2+2x(x+3)}{(x+3)(x-2)}$$

Simplifying leaves:

$$= \frac{x-2+2x^2+6x}{(x+3)(x-2)}$$
$$= \frac{2x^2+7x-2}{(x+3)(x-2)}$$

Equations

When algebraic fractions appear in equations, the fractions can often be removed by multiplying through by the denominator. For example,

$$\frac{x^2+4}{x+3} = x-7$$
$$x^2+4 = (x-7)(x+3)$$
$$x^2+4 = x^2-4x-21$$
$$4x = -25$$
$$x = -\tfrac{25}{4}$$

Exercises

39. Write as a single fraction

(a) $\frac{3}{x}+\frac{4}{x}$ (b) $\frac{x}{3}+\frac{x}{4}$ (c) $\frac{x}{3}+\frac{4}{x}$

(d) $\frac{3}{x^2}+\frac{4}{x}$ (e) $\frac{3}{x}+\frac{4}{x+1}$ (f) $\frac{3}{x-1}+\frac{4}{x+1}$

40. Simplify

(a) $\frac{x-1}{x^2-1}$ (b) $\frac{8x+16}{(2x+4)^2}$ (c) $\frac{3x-9}{x^2-9}$

(d) $\frac{12-4x}{2x^2-2x-12}$ (e) $\frac{2x^2-8}{x^2+x-12}$ (f) $\frac{x^3+3x^2+2x}{4x^2-16}$

41. Simplify

(a) $\frac{x^2+8x+12}{x^2-4}\times\frac{3x^2-18x}{x^2-36}$ (b) $\frac{x^2-6x+9}{x^2-x-3}\div\frac{x^2-9}{2x+4}$

42. Solve for x.

(a) $\frac{x+1}{x-1}=\frac{2x+5}{x+1}$ (b) $\frac{x+3}{x+1}=\frac{2x+3}{x+3}$

(c) $\frac{x+1}{x-1}=\frac{2x+8}{x+2}$ (d) $\frac{x}{2x+3}+1=x$

(e) $\frac{x}{x-6}+\frac{1}{x}=0$

Point geometry

You need to be able to:

- Find the gradient of a line segment.
- Find the midpoint of two given points.
- Find the distance between two given points.

Gradients

Gradient is $\dfrac{\text{change in } y}{\text{change in } x}$.

So, given two points with coordinates (x_1, y_1) and (x_2, y_2), the gradient is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Midpoints

The midpoint can be found by taking the average of the *x* values and the average of the *y* values.

So, given two points with coordinates (x_1, y_1) and (x_2, y_2), the midpoint is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

Distance

The distance between two points can be found using Pythagoras's theorem. You need to start by finding the difference between the *x* values and the *y* values.

So, given two points with coordinates (x_1, y_1) and (x_2, y_2), the midpoint is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1{}^2)}$.

For example, given points (1, 7) and (5, 3):

The gradient is given by

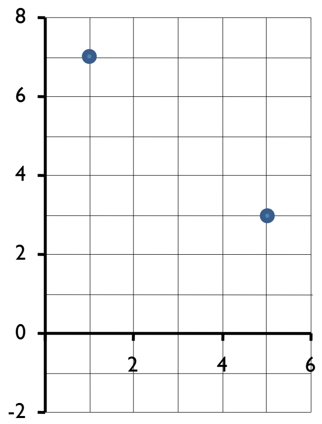
$$\frac{3-7}{5-1} = -1$$

The midpoint is given by

$$\left(\frac{1+5}{2}, \frac{7+3}{2}\right) = (3, 5)$$

The distance between the points is given by

$$\sqrt{(5-1)^2 + (3-7)^2}$$
$$= \sqrt{4^2 + (-4)^2}$$
$$= \sqrt{32}$$
$$= 4\sqrt{2}$$



Exercises

43. For each pair of points, find the gradient of the line segment they describe, the length of the line segment and the midpoint of the line segment.
    (a) $(4, 8); (9, -2)$
    (b) $(9, 5); (1, 1)$
    (c) $(3, 6); (-4, -8)$
    (d) $(105, -27); (-231, 37)$

Straight-line graphs

You need to be able to:

- Sketch equations of the form $y = mx + c$.
- Find the equation of a line given a point and the gradient.
- Use the fact that perpendicular gradients have product -1.

The equation of a line

A line with equation $y = mx + c$ has gradient *m* and *y*-intercept *c*. A gradient of 1 makes an angle of 45 degrees to the horizontal and runs SW-NE; a gradient of -1 makes an angle of 45 degrees to the horizontal and runs SE-NW. You need to know roughly what a gradient of 3 looks like, for example. Remember that gradient is $\frac{\text{change in } y}{\text{change in } x}$. A gradient of $\frac{1}{3}$ means that a line rises by 1 unit for every 3 it moves to the right.

Finding the equation of a line

If you are told the gradient and a point, or can work these out, you can easily find the equation of a line. Substituting in your known value of *m* and the point that you have been given will enable you to find *c*.

For example

Q: A line has gradient 3 and passes through the point $(3, 7)$. Find its equation.

A: We know that $m = 3$. Therefore $y = 3x + c$ and, using our known point:

$7 = 3 \times 3 + c$

$c = -2$

Since $c = -2$ the equation of the line is $y = 3x - 2$.

Parallel gradients

If two lines are parallel, they have the same gradient.

Perpendicular gradients

If two lines are perpendicular then their gradients multiply together to give -1.

For example

Q: Find the gradient of a line perpendicular to the line $y = 4x - 2$.

A: Since $y = 4x + 2$ has gradient 4, a perpendicular gradient (*m*) is such that $4m = -1$.

Therefore, $m = -\frac{1}{4}$.

The perpendicular line has gradient $-\frac{1}{4}$.

Exercises

44. (a) Find the gradient of the line through the points (3, 6) and (1, 14).
    (b) Hence find the equation of the line.

45. Find the equation of the line through the points (4, 20) and (–4, 8).

46. Find the equation of the line through the points (1, 7) and (4, 58).

47. Find the equation of the line through the points (1, 4) and (7, –2).

48. Find the equation of the line parallel to $y = 7x + 2$ through (5, 8)

49. Given the points: A (2, -1); B (3, 0)
    (a) Find the co-ordinates of M, the midpoint of AB
    (b) Find the distance AB
    (c) Find the equation of the line AB

50. A and B have coordinates (1, 4) and (5, 2)
    (a) Find the equation of the line AB
    (b) Find the perpendicular bisector of AB
    The perpendicular bisector of a line PQ passes through the midpoint of PQ at right angles to PQ

51. We wish to find the area of the triangle ABC where the points are defined as follows:
    A (-8, 0); B (8, -4); C (6, 5).
    (a) What is the equation of the line AB?
    (b) What is the equation of L, the line perpendicular to AB through C?
    (c) What are the coordinates of D, the point where L intersects AB?
    (d) Find the lengths AB and CD. Use these to find the area ABC.

Sketching quadratic and factorised cubic functions

You need to be able to:

- Use of the word root to describe intersection with the *x*-axis and *y*-intercept to describe intersection with the *y*-axis.
- Know that the minimum point on a quadratic lies half-way between the roots.

Quadratics and cubics

Remember that if a list of expressions multiplied together gives 0, one of those expressions must itself be 0. For example, if $(x+2)(x-7)=0$ then either $x=-2$ or $x=7$. Similarly, if $(x+2)(x-3)(x-8)=0$ then either $x=-2$, $x=3$ or $x=8$.

To sketch quadratics or cubics, it is useful to know where they meet the axes. You can find out where a curve meets the *x*-axis by letting $y=0$. You can find out where it meets the *y*-axis by letting $x=0$.

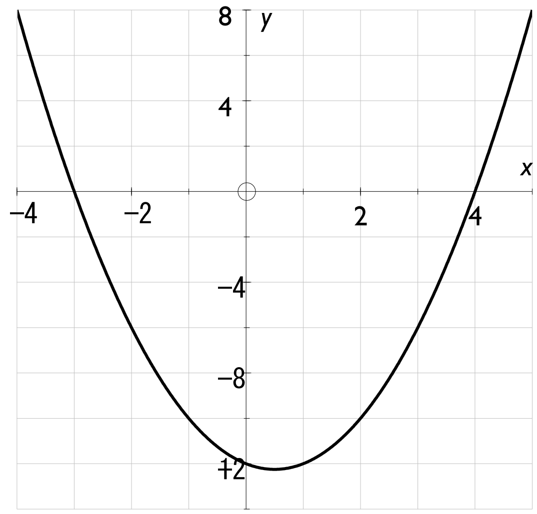
With a quadratic, the minimum point lies half-way between the roots (the *x*-intercepts).

For example:

Q: Sketch the graph of $y=(x+3)(x-4)$.

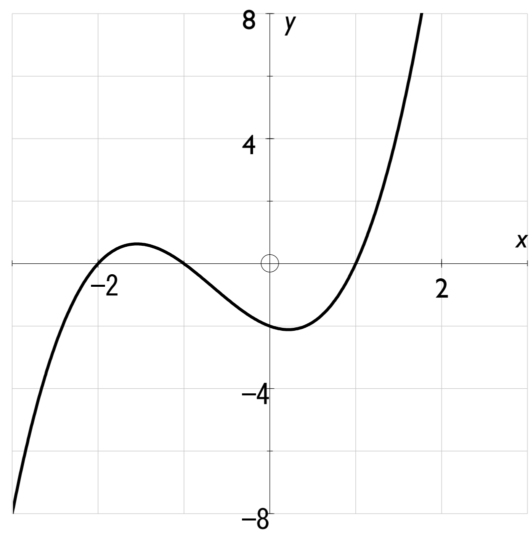
A: When $y=0$, $(x+3)(x-4)=0$. Therefore when $x=-3$ or $x=4$, this curve has $y=0$ and so passes through the *x*-axis.
When $x=0$, $y=(3)(-4)=-12$. Therefore, the curve passes through -12 on the *y*-axis.
Once we have drawn these three points, we join them with a smooth curve.

The minimum will lie halfway between the roots, i.e. when $x=\frac{1}{2}(-3+4)=\frac{1}{2}$.
The *y*-value will be $y=\left(\frac{1}{2}+3\right)\left(\frac{1}{2}-4\right)=-12.25$.



Q: Sketch the graph of $y=(x-1)(x+1)(x+2)$.

A: When $y=0$, $(x-1)(x+1)(x+2)=0$. Therefore when $x=1, -1$ or -2, this curve has $y=0$ and so passes through the *x*-axis.
When $x=0$, $y=(-1)(1)(2)=-2$. Therefore, the curve passes through -2 on the *y*-axis.
Once we have drawn these four points, we join them with a smooth curve.



Exercises

52. Sketch the following quadratic equations, marking points of intersection with the axes and the minimum or maximum point of the curve.
    (a) $y=(x+1)(x+3)$
    (b) $y=(2x+3)(x-4)$
    (c) $y=x^2+6x-7$
    (d) $y=x^2-8x-15$
    (e) $y=5+4x-x^2$
    (f) $y=x^2-5x-24$

53. Sketch the following curves, marking the points of intersection with the axes.
    (a) $y=x(2-x)(x+3)$
    (b) $y=(x+2)(x+1)(x+3)$
    (c) $y=(x-1)(x+4)(2x+1)$

Trigonometry

You need to be able to

- Use right-angled triangle trigonometry to find sides and angles
- Use the sine rule, including the ambiguous case
- Use the cosine rule
- Use the sine formula for the area of a triangle

Right-angled trigonometry

You need to be able to use the rules $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ and $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.

Labelling

We always label non-right angled triangles in the same way: angles with capital letters and sides with lower case letters. Angle A will be opposite a side of length *a*, and so on.

The sine and cosine rules

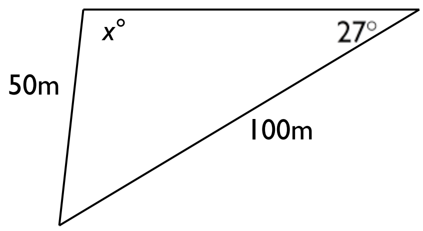
Do not use these on right-angled triangles!

You need to remember the sine and cosine rules: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ and $a^2 = b^2 + c^2 - 2bc\cos A$. You need to be able to apply them to find sides or angles.

When using the sine rule, sometimes you will need to find an obtuse angle. If this is the case, you must use the fact that $\sin(180-\theta) = \sin\theta$. This means that if your calculator tells you that the answer is 30, but you are looking for an obtuse angle, the real answer is $180 - 30 = 150$.

For example:

Q: Find the value of *x*, given that it is greater than 90°.



A: By the sine rule, $\frac{\sin 27}{50} = \frac{\sin x}{100}$. Hence $\sin x = \frac{100\sin 27}{50} = 0.9080$.
Using our calculator, we find that $\arcsin(0.9080) = 65.2°$.
Since our answer is obtuse, we find $x = 180 - 65.2 = 114.8°$.

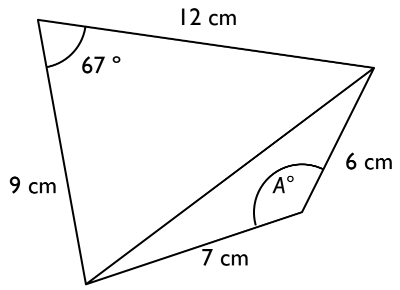
Areas

You need to know the area formula for a triangle, $\text{Area} = \frac{1}{2}ab\sin C$.

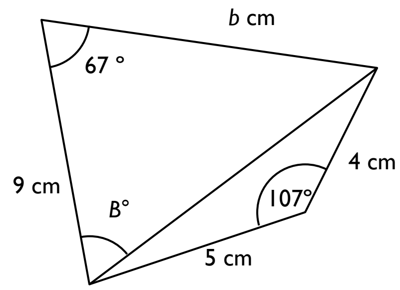
Exercises

54. Find the marked lengths and angles correct to one decimal place.

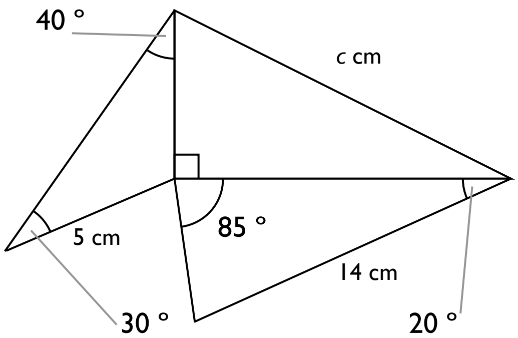
(a)



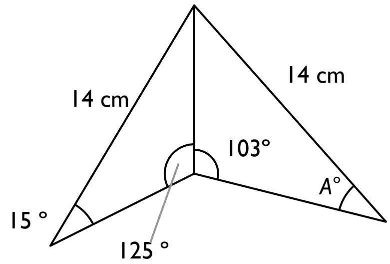
(b)



(c)



(d)

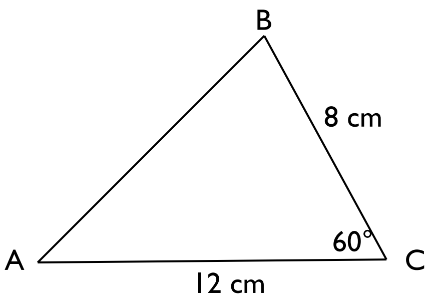


55. A church lies on a bearing of 030° from a point A. Bella starts 5km from A on a bearing of 040°. If Bella has to walk 4km to get to the church how far is the church from A? There are two possibilities.

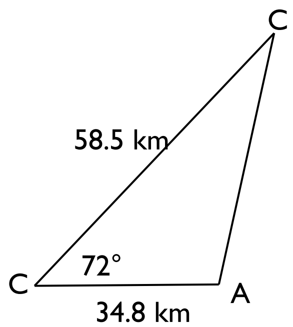
56. A boat leaves harbour and travels 9 miles on a bearing of 125°. A buoy is due east of the harbour and seven miles from the boat. Is it possible to say for certain where the buoy is? If it is, give the bearing from the boat to the buoy. If not, give all the possible bearings.

57. Using the result Area $= \frac{1}{2} ab \sin C$ find, to 3 significant figures, the area of each triangle:

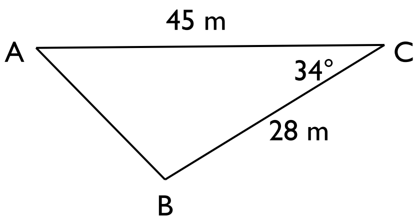
(a)



(b)



(c)



Lengths, areas and volumes

You need to be able to:

- Form expressions for lengths, areas and volumes of compound shapes.
- Solve equations based on these.

Shapes

You need to know all the formulae for the volumes and surface areas of common shapes. Some need memorising; some are common sense.

Sphere of radius r: Volume $= \frac{4}{3}\pi r^3$; Surface area $= 4\pi r^2$.

Cone of height h, base radius r: Slant height $= l = \sqrt{h^2 + r^2}$; Volume $= \frac{1}{3}\pi r^2 h$; Curved surface area $= \pi rl$

Cylinder of height h, base radius r: Volume $= \pi r^2 h$; Curved surface area $= 2\pi rh$.

Pyramid with base area b and height h: Volume $= \frac{1}{3}\pi bh$.

Equations

Often these expressions will need to be used to solve a problem. Remember to try to form an equation using the facts you have. Once you have an equation, all you need to do is solve it.

For example:

Q: A semicircle has perimeter 5 cm. Find its area.

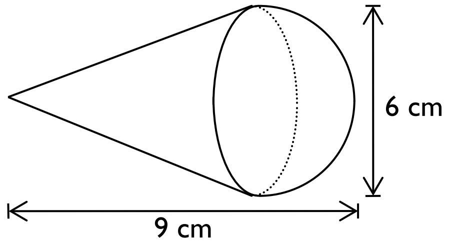
A: The perimeter of a circle is given by $2\pi r$, so the perimeter of the curved part of the semicircle will be πr. Therefore the total perimeter of the semicircle is $\pi r + 2r$. We can form an equation: $\pi r + 2r = 5$. Now we need to solve for r:

$$\pi r + 2r = 5$$
$$r(2 + \pi) = 5$$
$$r = \frac{5}{2 + \pi}$$
$$r = 0.972$$

Since we are looking for the area of the semicircle, we use $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi 0.972^2 = 1.49 \text{ cm}^2$

Exercises

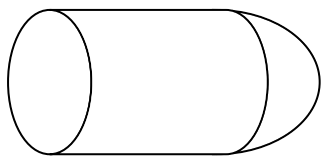
58. Find the volume of the illustrated shape, which is constructed from a cone and a hemisphere.



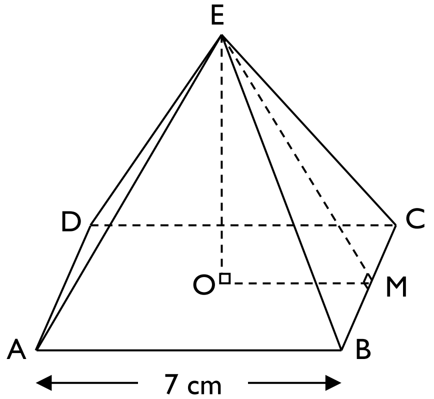
59. A square based pyramid is as tall as it is wide. If it has volume 9 cm^3, find its height.

60. A square based pyramid has base area 36 cm^2. The length of one of its diagonal edges is 5 cm. Find its surface area.

61. A bullet is made up of a cylinder and a hemisphere both of diameter 6 mm. If the total volume is 99π mm^3, find the height of the cylinder.



62. A square based pyramid of volume 98 cm^3 is pictured below. By calculating appropriate lengths find:
    (a) the volume of the pyramid;
    (b) the total surface area of the pyramid;
    (c) the angle that the sides of the pyramid make with the base of the pyramid.



63. Henry is trying to work out the volume of a football but he has remembered the formula for the volume of a sphere incorrectly. He thinks the volume is given by the formula $V = \frac{8}{5}\pi r^2$.
    He calculates that the volume of his sphere is 360π cm^3.

    Calculate the true volume of the sphere in cubic centimetres. You may leave your answer as a multiple of π.
    For what value of *r* would Henry's formula have given the correct answer anyway?